NMTC Problem-Set 1

Instructions: For this problem-set, it is assumed that you know the following topics to some degree. (An AoPs Volume 1 level knowledge suffices)

1. Logarithms
2. Series and Sums
3. Angle Chasing. Definitions of Incentre, Angle-bisector, and Orthocentre.
4. Basic Combinatorics

This problem-set is meant to be of the same level as the NMTC descriptive questions or a bit above that. Problems marked at the end with (*) are a bit more tough (and fun!).
These problems are meant to build your problem-solving skills, and as such, they are open-book. You are encouraged to use any textbook, video, or other online or offline resource as long as it doesn't give away the complete solution.
The sources to each applicable problem will be revealed with the solutions.

Problem 1: Miscellaneous

Consider a UFO on a number line starting at $x = 0$. Every second the UFO moves a constant, unknown distance d in the direction of the positive x-axis. For example, at time $t = 1$, the UFO will be at the coordinate $x = d$, at time $t = 2$, it will be at the coordinate $x = 2d$, and so on.
At any time t, you can check a random point on the number line, and see if the UFO is there or not. Prove that you can always 'catch' the UFO after a finite amount of time.
(Hint: There may be more than 1 answer)

Problem 2: Algebraic Manipulations

(a) There is a unique positive real number x such that the three numbers $\log_8(2x)$, $\log_4(x)$, and $\log_2(x)$, in that order, form a geometric progression with positive common ratio. Find x.

(b) Let x, y, and z be real numbers satisfying

$$\log_2(xyz - 3 + \log_5 x) = 5,$$
$$\log_3(xyz - 3 + \log_5 y) = 4,$$
$$\log_4(xyz - 3 + \log_5 z) = 4.$$

Find the value of

$$|\log_5 x| + |\log_5 y| + |\log_5 z|.$$

(c) Let $a > b$ be positive integers. Find the sum of all numbers which can be expressed in the form $\frac{1}{2^a 5^b}$.

Problem 3: Angle Chasing

(a) In acute $\triangle ABC$ with incenter I, prove that $\angle BIC = 90^\circ + \frac{1}{2}\angle BAC$

(b) In acute $\triangle ABC$ with incenter I. Let M be the midpoint of the arc BC not containing A. Show that M is the circumcenter of $\triangle BIC$

(c) In acute triangle $\angle ABC$, let I_A be the reflection of the incenter I over M (Defined as in the previous problem). Show that BI_A and CI_B are the external angle-bisector of $\angle ABC$ and $\angle ACB$ respectively.

(d) In acute $\triangle ABC$, Define I_B and I_C similarly to how I_A was defined earlier. Show that I is the orthocenter of $\triangle I_A I_B I_C$ (*)

Problem 4: Messy Algebra?

(a) Let m, and n be non-negative integers such that $n \geq m$ Prove that:

$$\sum_{k=0}^{m} \binom{m}{m-k} \binom{n}{k} = \binom{m+n}{m}$$

(b)

$$\binom{\binom{n}{2}}{2} = 3\binom{n+1}{4}$$

Prove that, for all $n \geq 3$, the above equality holds.

For both problems, there's an obvious method, and a correct method. (*)